A History and Philosophy of Algebra in Islamic Mathematics with a Focus on the Solution of the Polynomial

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Figure 1: Title cover of a copy of al-Khwarizmi's *Compendious Book on Calculation on Restoration and Balancing.*

1 Introduction

> "I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history."
>
> J. W. L. Glaisher
> English mathematician

From the outside, mathematics at higher education level could appear an exclusive club, alienating would-be mathematicians at an early age with many its axioms and nonsensical applied relevance or use. Many students may encounter further encumbrances to commit to a further education pathway when they find difficultly in recognising faces that resemble themselves within these circles of academia. Mathematicians must be given a blank slate to do their work, and after all, through the lens of mathematical rationale, when trying to solve complex problems, progress often results from diverse perspectives.

Teaching of mathematical history has extended this divide further and has brought out narratives about science that often focus on the singular, brilliant scientist, in most cases male and European, who makes substantial contributions through their innate genius. This is the expected outcome of a heavily western perspective on the history of mathematics in international academia. However, despite the ways in which the historical narrative is presented to the public, the evolution of mathematics is hardly as straightforward and linear as it may seem to be. We must therefore begin our delving into this history of mathematics as an attempt to unveil our own cultural prejudices. This will allow us to dissect our own ideas of mathematical thought to highlight the cultural links we share with the past and to question why other elements might seem alien to us.

As well as giving us a greater understanding of our discipline's history, it naturally leads us to a more elementary introspective inquiry into our own understanding of why mathematics is important and what use it has in a society. We should also, perhaps, start by asking ourselves what mathematics is to us today. Mathematician George W. Heine separates modern answers we give into three groups: pragmatic answers, which holds the emphasis of mathematics in the natural sciences, pedagogical answers, that states that the study of mathematics is useful for training the mind in abstract thought, and aesthetic answers, that shows that mathematical problems can be pleasurable pursuits.[1]

In this paper I will use medieval Islam to present what will hopefully be a fresh perspective on mathematical culture looked like in a time and society almost unrecognisable to us. My focus will be the academic lifetime and works of the Persian polymath Muhammad al-Khwarizmi in the seventh and eighth century.

[1] G. Heine, 'The Value of Mathematics - A Medieval Islamic View', in V. Katz (ed.), *Using History to Teach Mathematics: An International Perspective*, USA, The Mathematical Association of America, 2000, p.167.

2 Historical Background

We would be forgiven for thinking that ancient mathematics had solely sprouted from a practical necessity, yet in its long and messy history it not only dealt with practical use but with the abstract and the imaginary, often only limited by perspective. The great ancient civilisations constructed their own languages of mathematics, constantly producing new theories and methods in the pursuit of knowledge. The western-centric approach to modern science forces many mathematicians to fade out of this rich historical tapestry, and in its naivety attributes their given achievements to the men who came centuries after.

Students of mathematics, who throughout the world are taught under this western standpoint, would again be forgiven for thinking that the mathematical enlightenment during the European Renaissance of the 16th century was what finally matched the astonishing achievements of the ancient Greeks. This millennium is often disregarded as the Dark Ages, however this approach dismisses the fact that the international language of science during this time, for over 700 years, was Arabic. This alone burdens us with more questions we will have to attempt to answer. Why is a discipline that has not been in the west for most of history presented that way?

2.1 The "Golden Age" of Baghdad

The Golden Age of Islam refers to a period dated from the 8th to the 13th century, during which the history of Islam brought about a scientific, cultural, and economic flourishing. Ruled by various caliphates under this period, the science of the Islamic world bloomed under the reign of the Abbāsid tribe caliph Huran al-Rashid (c. 786-809 CE) in what was becoming the largest city in the world at the time, Baghdad.[2]

He inaugurated his Daar al-Hikma, or House of Wisdom, in an effort to begin an intellectual awakening after almost a century devoid of any worthwhile mathematical achievements. From the 8th century to the 14th century, the great intellectual centre of Baghdad brought together scholars familiar with Hellenic, Indian and Babylonian cultural and religious traditions that were commissioned to gather and translate the world's classical texts. Al-Rashid was seen as a patron of learning who united all these scholars under a singular government and common language of Arabic. Although still a subject of dispute amongst historians, the House of Wisdom's formal function as an academy is still one of debate complicated by a lack of physical evidence following the collapse of the Abbāsid tribe's caliphate. Unfortunately for archaeologists hoping to study the era, the contents of the Daar al-Hikma were destroyed in the Siege of Baghdad (1258 CE), leaving historians to piece together a narrative through translated texts and surviving biographical accounts. What is certain to us is that it was home to an extensive private library belonging to al-Rashid, which was then

[2] T. Chandler, *'Four Thousand Years of Urban Growth: An Historical Census'*, Lewiston, NY: The Edwin Mellen Press, 1987.

expanded to a public institute under his half-brother and successor Abu al-Abbas Abdallah ibn Harun al-Rashid (c. 813-833 CE).[3] This unique background created the perfect conditions for a new kind of mathematics to be formed that was more than a mere amalgamation of ancient traditions.

Figure 2: Depiction of an Arabic manuscript library. Maqamat of al-Hariri Illustration by Yahyá al-Wasiti, Baghdad 1237 *Bibliothèque nationale de France, MS Arabe 5847.*

Ancient astrologers, physicians, engineers, architects and mathematicians (most of which would have been seen as branches of mathematics rather than separate disciplines) began to bear new responsibilities with the spread of Islam and the Abbāsids rule. For instance, mathematicians were required to accurately determine times of prayer and the direction of Mecca, to track moon phases and codify calculations in Islamic finance. Named the qibla in Arabic, the sacred direction of prayer for Muslims faces the Kaaba, the building at the centre of Islam's most famous mosque, the Masjid al-Haram in Mecca, which most mosques are orientated toward. Although the dawn of the Islamic faith is said to have begun around 610 CE, this mathematical geography was not available to Muslims until the late eighth century. This is clear from the orientations of medieval mosques which are not accurately aligned according to the definition of the qibla.[4]

[3]S. Brentjes, R. Morrison *The Sciences in Islamic Societies*, The New Cambridge History of Islam, 4, Cambridge: Cambridge University Press, 2010, p. 569.

[4]D. King, *The Sacred Direction in Islam*, Interdisciplinary Science Reviews, Vol. 10, J W Arrowsmith, 1985, p.315.

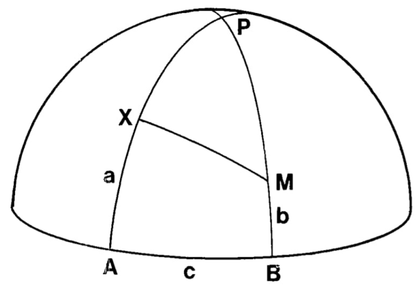

Figure 3: The problem of finding the direction of Mecca M from a point X requires find the angle which XM makes with PXA. (*D. King, The Sacred Direction in Islam.*)

Although founded on the Islamic principles of education through various Quranic injunctions and Hadith, the academic culture exhibited an incredible degree of tolerance. Showing a strong interest in assimilating knowledge through classic works outside of their heritage, this inclusivity even extended to theology.[5] Muhammed ibn Zakariya al-Razi (865-925), more commonly known by his Latinized name Rhazes, was a notable polymath of the Islamic Golden Age who wrote:

> *'Books on medicine, geometry, astronomy, and logic are more useful than the Bible and the Qur'an. The authors of these books have found the facts and truths by their own intelligence without the help of prophets.'* [6]

Many Muslim scholars associated with the House of Wisdom were leaders of their field from the eighth to the 14th century. In fact, many mathematical studies in the medieval Latin Europe were stimulated by Latin translations from the Arabic. One of the most important mathematical sources developed from the House of Wisdom was the treatise on solution of polynomial equations written by Muhammad al-Khwarizmi (825 CE).

2.2 Muhammad al-Khwarizmi

The most important ninth century mathematician, often described as the Father of Algebra, Muhammad Ibn Musa al-Khwarizmi (or Muhammad son of Moses from Khwarizm) was one of the most esteemed members of the House of Wisdom. The Persian travelled to Baghdad in the early ninth century from a region of Central Asia just south of the Aral Sea, in modern-day Uzbekistan.

[5] H.N. Rafiabadi, (ed.), Challenges to Religions and Islam: A Study of Muslim Movements, Personalities, Issues and Trends, Part 1, Sarup & Sons, 2007, p. 1141.

[6] A. Bawadi, *Muhammad ibn Zakariya al-Razi* in M. Sharif (ed.), *A History of Muslim Philosophy*, Otto Harrassowitz, 1963], chapter XXII, pp421-433

Brought up as a Zoroastrian, many historians believe that he subsequently converted to Islam. In his pioneering work, *al-Kitāb al-Mukhtasar fī Hisāb al-Jabr wal-Muqābalah*, 'The Compendious Book on Calculation and Restoration', he opens with the line *'Bism-Illāh al-Rahmān al-Rahīm'* ('In the name of God, the most Gracious and Compassionate'), the phrase recited before the reading of the Quran with which a great amount of books written by devout Muslims begin, even to this day.

Figure 4: An imaginary portrait for al-Khwarizmi derived from a soviet stamp.

Muhammad Al-Khwarizmi produced incredibly influential works in both mathematics and astronomy and was soon appointed as the head of the library and academy of the House of Wisdom. He was first and foremost known as a mathematician; his treatise on Hindu-Arabic numerals not only introduced the Muslim world to the decimal place-value system, but through the Latin translations of his work during and after the 12th century, his work contributed to the eventual diffusion of these ideas into European mathematics.

The tabulation of the coordinates of hundreds of cities in the known world and instructions for drawing a better map of the world in his famous *Kitab surat al-ard* (Picture of the Earth) treatise secured his legacy as the first geographer of Islam, whilst his astronomical work conducted at the Shammāsiyya observatory marked him out as one of the world's great astronomers. Yet all these achievements pale alongside his greatest claim to fame, which is without doubt his book on what we now describe as algebra. In fact, the word algorithm originates from the latinization of his name, *Algoritmi*, thanks to his organisation in his description of procedures.

Whilst his fame is well deserved, especially on the restoration and reduction elements of algebra, it is only fair to say that his methods for finding unknown quantities were based on a number of ancient traditions.

3 A Condensed History of Algebra Up Until the House of Wisdom

3.1 Historical Stages of Algebra

Studying the science produced from the House of Wisdom can hardly be limited to the efforts made under Islamic rule. The development of algebra up until that time period is as important as the produce of the Muslim scientists' minds. It was the amassed knowledge of the ancient world that they used to construct such wonders in mathematics and engineering. Here we will see the efforts of mathematicians across the centuries that eventually lined the shelves of the House of Wisdom. This also allows us to have a general understanding of where our protagonist al-Khwarizmi lies in the timeline of this branch of mathematics.

Historians often consider the history of algebra to be separated into two significant parts: "classical algebra", that was primarily concerned with the solving of polynomial equations and "modern algebra", which is devoted to studying algebraic structures. The latter, also referred to as "abstract algebra" is largely a product of the 19th and 20th centuries. The stage of classic algebra can be broken into several distinct phases of historical thinking. These are not limited to certain societies or regions but mostly encompass broad eras of mathematical knowledge and understanding. The conceptual stages of algebra as described by historian Victor J. Katz are as follows:[7]

- **Geometric stage**, where geometric methods are used as the core concepts of algebra. This stage originates with the Babylonians through to the ancient Greeks, and was later revived for higher order calculations by Omar Khayyám.

- **Static equation-solving stage**, where the purpose is to find numbers satisfying particular relationships. Algebra did not decisively progress to the static equation-solving stage until al-Khwarizmi introduced generalized algorithmic processes for solving algebraic problems however this divergence from geometric algebra is best attributed to the likes of mathematicians Diophantus and Brahmagupta.

- **Dynamic function stage**, where motion is a core underlying idea. The idea of a function was first proposed by Persian mathematician Sharaf al-Dīn al-Tūsī (c. 1135-1213), but the lack of explicit functions with symbols meant that algebra did not decisively move to the dynamic function stage until Gottfried Leibniz.[8]

[7] V.J. Katz, B. Barton, *Stages in the History of Algebra with Implications for Teaching*, Educational Studies in Mathematics, Vol 66, No. 2, 2007, p. 185–201.

[8] P. Nasehpour, *A Brief History of Algebra with a Focus on theDistributive Law and Semiring Theory* ,Golpayegan University of Technology, Iran, 2018, p. 2.

Another perspective one might take is that of the evolution of the sub-stages of development of notation from the rhetorical to the symbolic. Many authors of historical research have also recognized four noteworthy stages of algebraic notation in the development of algebra alongside the evolution in conceptual understanding:

- **Rhetorical algebra**, in which equations are written in verse. First developed by the ancient Babylonians, this era of expression continued up until the 16th century. For example, the rhetorical form of $x + 2 = 4$ is "The thing plus two equals four". In the later years numbers might have been replaced with the decimal numeral system.

- **Syncopated algebra**, the stage which blends ideas from the rhetorical and symbolic in which some symbolism is used but does not contain all of the characteristics of symbolic algebra. Syncopated algebraic notation first appeared in Diophantus's *Arithmetica* (3rd century AD), followed by Brahmagupta's *Brahma Sphuta Siddhanta* (7th century). Usually presented with restrictions in the use of core axioms or operations familiar with us today, the understanding of algebraic objects and methods are not in line with that of symbolic algebra.

- **Symbolic algebra**, in which full modern symbolism is used along with the characteristics of abstract algebra. An early concept of this can be found in the work of Islamic mathematician Ibn al-Banna (13th-14th centuries) aptly named *Raf al-ijāb*, 'Lifting the Veil', covering topics such as continued fractions and square roots. This algebraic notion was further developed by Abū al-Hasan al-Qalasādī in the 15th century. Full symbolic algebra was developed by François Viète (16th century) and later expanded by René Descartes (17th century), introducing the modern notation and linked geometric problems to algebraic expression with Cartesian geometry.[9]

[9] C. Boyer, *Revival and Decline of Greek Mathematics*, 2nd Ed., New York, Wiley, 1991, p.180.

3.2 Babylonian Era Algebra

Babylonian mathematics was largely based on the Sumerians' numeric system, a people from 4000 BCE in modern day southern Iraq. The Babylonians marked the start of the first conceptual stage of algebra, where the concepts of algebra are laid out in geometric form.

The Babylonians relied on a positional sexagesimal system (system of base 60) in contrast of our modern decimal system of base 10. The fact that we divide hours into 60 minutes and minutes into 60 seconds is due to the mathematics of Mesopotamian civilisations. A common theory for why this was states that the Sumerians were a product of a merging of two earlier peoples with number systems of bases 5 and 12, creating a common system on 60 for mutual understanding, however this might not be the case since the Babylonians relied on a system of six groups of ten, not five groups of twelve.[10] Although from our perspective, this system is an over-complication from our simple modern system of base 10, the Babylonians made no consistent use of zero in their mathematics.

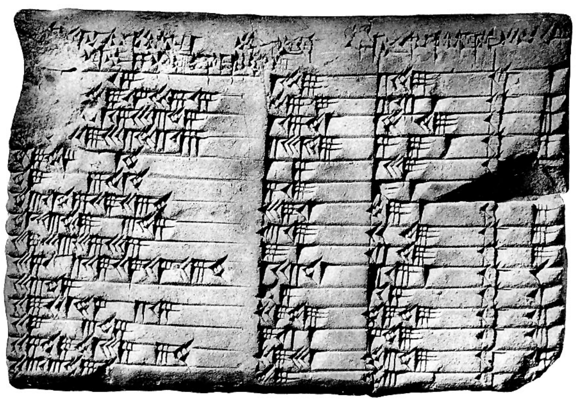

Figure 5: Plimpton 322 is a Babylonian clay tablet, notable as containing an example of Babylonian mathematics. It has number 322 in the G.A. Plimpton Collection at Columbia University.

The earliest evidence of written algebraic writing is that of Old Babylonian cuneiform texts on clay tablets. Most of their mathematics was conducted within tables including multiplication, squares (not cubes), and square and cube roots (Figure 5).[11] As well as tables, the Babylonians frequently wrote problems that attempted to find the solution of an unknown number. For example, an ancient Babylonian problem would take the form: *'What is the number, when added to its reciprocal, gives a known number?'* Historians class this sort of

[10]S.L. Macey, *The Dynamics of Progress: Time, Method, and Measure*, Atlanta, Georgia: University of Georgia Press, , 1989, p. 92.

[11] *"158. Cuneiform Tablet. Larsa (Tell Senkereh), Iraq, ca. 1820-1762 BCE. – RBML, Plimpton Cuneiform 322"*, Jewels in Her Crown: Treasures of Columbia University Libraries Special Collections, Columbia University, 2004.

statement in which equations are written in verse as the dawn of rhetorical algebra. First developed by the ancient Babylonians, this era of notation remained dominant up until the 16th century.

With our modern approach we would write the unknown number as x and the known number as b. We can then express this as an equation:

$$x + \frac{1}{x} = b.$$

Rearranging the equation into the more familiar form produces $x^2 - bx + 1 = 0$. For this particular example it is of the form:

$$x = \frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - 1}.$$

This informs us that, if given the value of b, you can work out x. The Babylonians knew this idea outside of our formulation having written entirely in prose, as did the Greeks. Other problems presented a concrete problem to be solved, such as the partitioning of a field amongst a certain number of sons, rather than providing a general algorithm for any similar problems.

Although the Babylonians and the Greeks pre-date the 'Golden Age of Islam' in terms of methods of solution, a huge portion of al-Khwarizmi's work does not emerge in the course of solving a particular problem but rather provides the algorithm to solve an infinite class of problems. The concept of an equation for its own sake, although in prose, is laid out at the start; an exposition beginning with primitive terms that henceforward lays down an explicit algorithm to follow. The abandonment of attempting to solve a particular problem to provide a rhetorical 'algorithm' is what sets him apart from these very similar texts. It is a contrast from some Greek sources that supplied various specific examples, leaving the reader to assume that it can apply to other similar circumstances; general solutions were written out but not in general terms.

3.3 Ancient Greek Geometry and Euclid's *Elements*

The path down which history has taken ancient mathematical ideas must pass through a dead museum we think, unlike the works of familiar European mathematicians that fill our textbooks and brief introduction to modules that have stood the test of time. If only a handful of figures have been taught to us as founders of fresh ideas of their time, then it is only natural to conclude that ancient natural philosophy is simply just exactly that, ancient. This ancient path however carries through ideas which are very much alive and have aided scholars across the world in their quest to expand mathematical knowledge.

In the scientific realm of the ancient Greeks, scriptures were cited as the final authority for the truth of a mathematical proposition. There was no universal standard of truth other than tradition. If any disagreement arose in the interpretation of a verse of text, there was neither a test nor proof for the veracity of a given statement. It is, perhaps, the detail that gave these texts some verisimilitude. This way of thinking extended universally across the ancient world. The Greek natural philosophers contributed to the philosophy of science their way of thought rather than their intellectual rigour.

It was Euclid's *Elements* (300 BCE), arguably the apotheosis of Greek scientific thinking, in which a new academic standard was introduced; his famous work on geometry paved the way for an abstraction of the modern scientific method introducing a new test and authority for the substantiation of truth. He had his own theory involving second-order equations that was laid out through the façade of geometric theorems. First translated by al Hajjaj ibn Yusuf, a contemporary of al-Khwarizmi and notable governor of the Umayyad Caliphate, under the reign of al-Rashid, the Arabic translation of the *Elements* was further improved during the period of the seventh caliph al-Ma'mun's era House of Wisdom.

A familiar problem for mathematicians to test their understanding of equations of the second degree can be written using Euclid's *Elements* (Book Two, Proposition Eleven): *'Divide the straight line AC, which is of known length, into two unequal segments: AB and BC. What are the lengths of these segments such that a square of side AB will have the same area as a rectangle of sides AC and BC?'* This problem is an example of a requirement of the solution for positive real roots of a quadratic equation solved geometrically. Around the same time an attempt was made by the Greeks to seek out a geometric construction for the solution of the cubic by expanding the work in the *Elements*. This endeavour was in vain as it is now known that the general cubic cannot have a solution with the use of Euclidean tools.[12]

As the most coherent work on geometry the world had ever seen, the *Elements* soon came to be a standard text that was used in schools across the globe up until the turn of the 20th century. Mathematics however required a far neater and more efficient way of transposing this question to set up the problem algebraically.

[12] C.D. Smith, *Developments in Secondary Mathematics.* Mathematics News Letter 5, no. 4, 1930, p. 4–8.

When mathematicians express Euclid's geometric constructions in our modern algebraic language, they give birth to algebraic identities and solve quadratic equations. This symbolism would have been completely alien to the Greeks however and was never used in their work. There were no such thing as general symbolic operations in anything resembling abstract language and hence they join the Babylonians in the rhetorical algebra stage of notation. This similarity with the Babylonians extended to the definition of numbers. Zero and negative numbers were not considered numbers since Euclid's stipulation was that a number was a collection of units. Even the understanding of the number one was ambiguous since it did not fall under the stipulation of a collection.

Several historians state that there is not a clear answer as to the question of whether the Arabic translations of *Elements* would have been available to al-Khwarizmi. The Persian polymath dedicated one chapter of his book solely to geometry called Bāb al-Misāha or the 'Door to Geometry', however the fact is that there are no direct references to Euclid's *Elements*. Although he makes use of a particular proposition from Book II, he has neither definition, nor axioms nor any geometric presentation of the direct Euclidian kind. The Door to Geometry chapter is purely one of mensuration, a compilation of rules for the use of land surveyors.

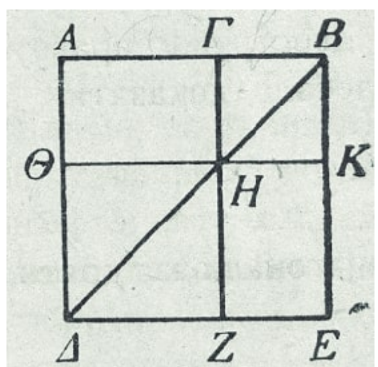

Figure 6: A diagram taken from a copy of Euclid's Elements Book II Proposition 4.

Modern historians would argue that it is the accompaniment of a geometric presentation alongside his recipe for algebraic solution in another section of his treatise that suggests the awareness and the inspiration of Euclid's *Element*s. Whatever the truth, historians of mathematics will agree that those aspects of his work on algebraic methods that cannot be classed as original thought are not important features. The diagrams showing the completion of the square that date back to Babylonian times for example are only used as a means of justifying the solutions achieved via algebraic methods.

3.4 Diophantus's *Arithmetica*

Hellenistic mathematician Diophantus lived in Alexandria (Egypt) between 150 and 250 CE. Very little is known about his life other than the fruits of his fantastic mind. His multi-volume manuscript (the surviving content has since been subdivided into nominal 'books', but these don't necessarily correspond to physical volumes) *Arithmetica* pre-dates our Persian mathematician by about five centuries although translations of Diophantus' works were not made available in Arabic until several decades after *The Compendious Book on Calculation and Restoration*. Divided into thirteen books, only ten survived with six in the original Greek and four thought to have been translated from Greek to Arabic by the Syrian Christian philosopher and mathematician Qusta ibn Luqa (820-912). Historian of mathematics Norbert Schappacher has written:

> *'[The four missing books] resurfaced around 1971 in the Astan Quds Library in Meshed (Iran) in a copy from 1198 AD. It was not catalogued under the name of Diophantus (but under that of Qusta ibn Luqa) because the librarian was apparently not able to read the main line of the cover page where Diophantus's name appears in geometric Kufi calligraphy.'*[13]

Arithmetica in the third century tackled a wide range of mathematical problems including multiplying positive and negative terms, simplifying quantities and use of different powers of the quantity. Although the title describes what we would today class as a separate branch of mathematics, this higher-order perspective on arithmetic can be recognised in the modern day as algebraic thinking. Several historians will attribute Diophantus's mathematics as the transition from the geometric to the static equation-solving stage, where the objective is to find numbers satisfying certain relationships, however algebra did not decisively move to the static equation-solving stage until al-Khwarizmi introduced generalized algorithmic processes for solving algebraic problems. The work was the first to include abbreviations to represent powers, operations and unknowns, thus using what is now known as syncopated algebra. The main difference between that and our modern notation is that syncopation lacked a specific set of symbols for operations and relations. For example, from modern notation we would have:

$$x^3 - 2x^2 + 10x - 1 = 5$$

Rewritten for a symbol-for-symbol translation as:

$$(x^3 1 + x10) - (x^2 2 + x^0 1) = x^0 5$$

[13] N. Schappacher, *Diophantus of Alexandria : a Text and its History (PDF)*,Institut de Recherche Mathématique Avancée, 2005, p. 18. Retrieved August 2021.

Would be written in Diophantus's syncopated notation as:

$$K^{\upsilon}\overline{\alpha}\,\zeta\overline{\iota}\;\;\pitchfork\;\;\Delta^{\upsilon}\overline{\beta}\,\mathrm{M}\overline{\alpha}\;\text{ἴσ}\;\;\mathrm{M}\overline{\varepsilon}$$ [14,15]

Symbol	Representation
$\overline{\alpha}$	1
$\overline{\beta}$	2
$\overline{\varepsilon}$	5
$\overline{\iota}$	10
ἴσ	equals
$\pitchfork$	subtraction
M	the zeroth power
ζ	the "first power"
Δ^{υ}	the second power
K^{υ}	the third power

Diophantus provided detail on solving equations using restoration and balancing, the titular techniques of al-Khwarizmi's text, using these methods to solve polynomials up to the 6th degree. He was the first to work on problems that would have multiple if not an infinite set of solutions. Rooting his equations in geometric problems however meant that his work did not progress to a radical new branch of mathematics.

Most famously, equations involving two or more unknown values raised to any power, such that the solutions are always integer numbers are called Diophantine equations, although he himself did not provide any general methods of solution (the manipulation of these equations gave rise to the term Diophantine analysis). He also allowed positive rational numbers as possible solutions, not just integers. This was as far as his mathematical tolerance extended since he called a problem whose only solutions were negative "absurd".[16]

Ultimately, Diophantus was mostly interested in number theory: numbers and the relationships between them, whilst Euclid's interest lay in the geometric branch of this mathematical bush. Similar to the works of Indian mathematician Brahmagupta, *Arithmetica's* scope of interest was broad and covered a wide range of various modern branches of mathematics. The Muslims that came after them employed a clear-cut argument from problem to solution in an organised, systematic fashion – respects in which neither Diophantus nor Brahmagupta excelled.

[14] J. Derbyshire, *Unknown Quantity: A Real And Imaginary History of Algebra*, Washington, DC, Joseph Henry Press, 2006, p. 35-36.

[15] R. Cooke, *The History of Mathematics: A Brief Course*, Wiley-Interscience, 1997, p. 167-168.

[16] The most famous Diophantine equation is that highlighted by the famous Pierre de Fermat in his personal copy of Claude-Gaspard Bachet's 1621 edition of the Arithmetica. It states that there are no integer values for x, y and z such that xn+yn=zn, when n is greater than 2. *"It is impossible to separate a cube into two cubes or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain".*

3.5 Brahmagupta

Indian mathematician and astronomer Brahmagupta of Gujrat (c. 598-670 CE) developed very precise procedures for linear and quadratic equations with multiple variables. Despite the fact that numbers were written out in words as opposed to symbols, his work, *Brahmasphutasiddhanta* was the most advanced number theoretic text the world had ever seen. Within this widely influential text, he presented the general form of a quadratic equation and even included a technique for solving Pell equations (any Diophantine equation of the form $x^2 - ny^2 = 1$).

His work also consisted of the first truly definitive remark on the number zero. He stated that zero was the result of combining a credit and debit of equal value and also defined several axioms such as one plus zero is one, one minus zero is one and one time zero is zero. These axioms defined an important step in algebra since they defined zero as a number in its own right. Brahmagupta named this number 'sunya' which meant empty or space. When the Arabs made use of Indian numeration in the tenth century, the name for zero was translated to 'sifr', the Arabic word for empty.

A great interest for the Islamic mathematicians was the reception and transmission of Indian ideas on numeration to which decimal fractions were added. In fact, records from Said-Al-Andali, an Arabic scholar from the time, show that al-Khwarizmi was familiar with Indian works on number theory. In his book *Tabaqat al-'Uman* (Categories of Notation) he informs us that al-Khwarizmi had personally had a hand in translating work from that of Brahmagupta, a renowned Indian mathematician. Yet this close familiarity of the computational portions of Brahmagupta's work seems to be in question since no Arabic scholars made use of negative numbers or of the syncopation found in his work.

4 The Compendious Book on Calculation by Restoration and Balancing

4.1 Legacy

Figure 7: Title page of a copy of The Compendious Book on Calculation by Restoration and Balancing.

Al-Khwarizmi's book on the solution of linear and quadratic equations is regarded as the foundation and the cornerstone of this branch of mathematics, eventually granting him the grand title of Father of Algebra. *al-Kitāb al-Mukhtasar fī Hisāb al-Jabr wal-Muqābalah*, 'The Compendious Book on Calculation by Restoration and Balancing', was the treatise that cemented algebra as a discipline in its own right, collating various different algebraic methods to form algorithmic approaches to a wide range of problems.

It was in fact the Latin translation of his book title, *Liber algebrae et almucabala*, which resulted in the name 'algebra' being applied to the theory of equations, with the definition expanding to the much wider abstract sense we know today. This text, although containing little mathematical concepts that were original, was widely used in both Arabic and Latin and was regarded as the defining reference for linear and quadratic equations. Whilst the use of the word is linguistically anachronistic in that its origins are from the ninth century, most people asked today would say its core use would refer to the school subject in which equations are solved. When attempting to describe the mathematics before this period of enlightenment, historians have also extended the use of the word to refer to manipulation of abstract quantities according to (sometimes contentious) rules such as 'Greek geometric algebra' and 'Babylonian algebra'.

What separated algebra from the geometry or arithmetic in the years of Islamic scientific development was this idea of a broader concept and unifying theory, different notions to that of Greek writers, that had not happened before. It was quite a distinct move away from the ancient Greek idea of mathematics which by today's definition was primarily geometry. The Islamic 'al-jabr' allowed rational numbers, irrational numbers, and geometrical magnitudes and more to all be treated as 'algebraic objects'. Given the ubiquitous mathematical problems in society that needed to be solved algebraically, whether involving working out areas of land for agriculture, calculating inheritance or taxes, it would hardly be surprising that 'algebra' in some form existed long before the Golden Age of Islam. The question as to when algebra as a new branch of mathematics originated relies on our definition of algebra itself.

4.2 A Compilation and Extension of Existing Rules

We see how 'algebra' is derived from 'al-jabr'. There are several literal translations of the title of the treatise (concerning *al-jabr* and *al-muqabala*). The former (completion or restoration), which was also used in English for a time for the setting of a broken bone, denotes the transference of a negative quantity from one side of the equation to the other. The latter (balancing or opposition) as the adding of the same term on the same side. Repeated application of this rule makes quantities of each type ('square','root','number') appear in the equation at most once, which helps to see that there are only six basic solvable types of the problem.
His work is split into three parts:

1. An exposition on the rules for solving equations, followed by 40 sample problems.

2. A section on mensuration.

3. Practical uses of the Islamic rules of inheritance.

The importance of this final section (that is half the book) is shown by the emphasis given in al-Khwarizmi's abstract: [17]

> *'A short work on calculating by al-jabr and al-muqabala, confining it to what is easiest and most useful in artithmetic, such as men constantly require in cases of inheritance, legacies, partition, lawsuits, and trade, and in all their dealings with one another, or where the measuring of lands, the digging of canals, geometric computation, and other objects of various sorts and kinds are concerned.'*[18]

The treatise begins with a brief outline of the decimal numbering system and then in six chapters presents the six types of equations which illustrate the elementary operations in which linear and quadratic equations are handled. Thus, the equations are verbally described in terms of 'squares' (what would today be 'x^2'), 'roots' (what would today be 'x') and 'numbers' ('constants':

[17] D.A. King, "Mathematics applied to aspects of religious ritual in Islam". In I. Grattan-Guinness (ed.). *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, 1st Ed., JHU Press. 2003, p. 83.

[18] A. Jamil, *The Determination of the Coordinates of Positions for the Correction of Distances between Cities - a Translation from the Arabic of al-Biruni's Kitab Tahdid Nihayat al-Amakin Litashih Masafat al-Masakin*, American University of Beirut, 1967.

ordinary spelled out numbers, like 'forty-two'). It is important to note, however, that only positive numbers were considered true solutions to equations. The six types, with modern notations, are:

- squares equal roots ($ax2 = bx$)
- squares equal number ($ax2 = c$)
- roots equal number ($bx = c$)
- squares and roots equal number ($ax2 + bx = c$)
- squares and number equal roots ($ax2 + c = bx$)
- roots and number equal squares ($bx + c = ax2$)

In the second section of his book, propositions from Book II of Euclid's Elements are used in order to accompany his algorithms with geometric examples. Al-Khwarizmi was in fact the first to make this geometric justification for the solution of the quadratic from these elementary propositions.

Figure 8: A page from a copy of 'The Compendious Book on Calculation by Restoration and Balancing' showing a geometric justification.

One of the core elements that define algebra is the use of the unknown. When al-Khwarizmi describes his unknown object, there is no specific allocation of value. Thus, although written in prose, al-Khwarizmi is in fact closer to the algebra we know of today in our use of the unknown quantity. The thing or 'shay' in Arabic was no longer simply a placeholder for numbers, but a newly created algebraic object that was able to be manipulated in its own right. The 'shay' was an entity ready to have the many rules of 'al-jabr' enacted on it.

5 The Algebra after al-Khwarizmi

5.1 The Continuation of Islamic Mathematics

Many Muslim scholars continued the development of algebra both in the House of Wisdom and beyond following al-Khwarizmi's death in 850 CE. Arguably on whom his work had the most influence on was the Egyptian mathematician Abu Kamil Shuja ibn Aslam (c. 850–930 CE) who became the first mathematician to accept irrational numbers as solutions and as coefficients in an equation.[19] Writing his *Algebra* around 900 AD based on al-Khwarizmi's, he was also the first to find a solution for three non-linear simultaneous equations with three unknown variables.[20] Approaching the solution of a quadratic equation as a number rather than a geometric representation, along with making use of the increasingly popular decimal system, Abu Kamil's approach pushed for the cementing of a general abstract idea of a number in mathematics, essential for the subsequent creation of an full equation as we know today.

Another great contribution comes from the 10th-century Persian mathematician and engineer Al-Karaji (c. 953-1029 CE). Perhaps associated with the most recognisable shift in algebraic notation, he replaced geometric operations with arithmetic ones. His work on algebra included rules taken from al-Khwarizmi's work on manipulating polynomials but extended much further with his enquiry into binomial coefficients. He was also a mathematician that investigated Pascal's triangle (along with Indian mathematicians centuries before Pascal).[21] Al-Karaji's work could be seen as the inspiration for the beginning of the dynamic function conceptual stage that would dawn a few hundred years down the line of the Islamic Golden Age.

Omar Khayyam (c. 1050–1123 CE) was a noteworthy Persian polymath who's work on astronomy and geometry solidified Islam's Golden Age. His 1070 CE work *Risālah fil-barāhīn alā masāil al-jabr wal-muqābalah*, 'Treatise on Demonstration of Problems of Algebra', made reference to Greek knowledge of intersecting conic sections to problems concerning cubic equations. His approach of methods of solution were the same as many other Islamic mathematicians - he provided the systematized axioms and application of them with the geometric justification. Within this work he included his research on the triangular array of binomial coefficients (Pascal's triangle). Seven years later, he published his book on non-euclidean geometry entitled *'Sharh ma ashkala min musadarat kitab Uqlidis'* meaning 'Explanations of the Difficulties in the Postulates of Euclid'. This was later translated from the Arabic to *'On the Difficulties of Euclid's Definitions'*. Apart from his scientific pursuits, he had also gained fame through the literary circles with his poems. Finally translated into English in the 1800s, he wrote over a thousand verses or 'rubaiyat'.

[19] J. Sesiano, 'Islamic mathematics' in H. Selin(eds.), *Mathematics Across Cultures: The History of Non-Western Mathematics*, Springer, 2000, p. 148.

[20] J.L. Berggren, *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook.* Princeton University Press, 2007, p. 518.

[21] J.L. Coolidge, *The story of the binomial theorem*, The American Mathematical Monthly, Vol. 56, No. 3, 1949, p. 147–157.

5.2 Late Middle Ages and Modern Algebra

The dawn of the twelfth century saw a new wave of Islamic knowledge flowing into Europe through translations made in the Crusader states, Italy and Spain. Seen as an "academic" enterprise before the European Renaissance, Islamic mathematics had little use in daily work in commerce or construction in Italy up to the 13th and 14th centuries. Al-Khwarizmi's book on algebra was translated into Latin in 1145 as *Liber algebrae et almucabala* by the Englishman Robert of Chester and around a similar time by the Italian Gerard of Cremona.

Leonardo of Pisa, commonly known as Fibonacci, was instrumental in introducing the West to 'modus Indorum' (translated to 'method of the Indians') or the Hindu-Arabic numeral system through his book *Liber abbaci* (1202).[22] In his preface he makes use of the Latin term 'algorismus' describing al-Khwarizmi's writings on Indian calculation, yet Fibonacci only makes mention of the first name of al-Khwarizmi once in Chapter 15, using the Latin 'Maumeht', possibly to give the impression that the bulk of his work was innovatory.[23]

Several Italian mathematicians furthered the mathematical knowledge by adding numerical solutions for cubic and biquadratic equations which gradually developed into the study of the character of the roots. Descartes introduced mathematical notation and along with Girard concluded that an equation of n degree can have no more than n roots. The fundamental theorem of algebra was proven by Gauss in 1799.

[22] R.E. Grimm, *The Autobiography of Leonardo Pisano*, Fibonacci Quarterly, Vol. 11, No. 1, 1973, p. 99–104.

[23] L.E. Sigler, *The English Edition of Fibonacci's Liber Abaci*, New York, Springer-Verlag, 2002.

6 Conclusion

> "One can conceive history as an argument without end."
>
> Pieter Catharinus Arie Geijl
> Dutch historian

Mathematicians can often succumb to the idea that mathematics is a unified ahistorical entity guided by universal rationality, yet we can see that even from this very brief exploration of the development of algebra that our discipline is a diverse process. It is not separate from the historical context in which it lies; it is in fact guided by various interests, beliefs and values in a given time and place in the world. Researching history as a mathematician can feel like attempting to describe a fractal - you can achieve a great sense of understanding of a culture or concept in any one frame but there is always a great amount of scope within that can be given to create as close to an accurate picture as possible.

Perhaps it is in our tunnel vision of the sciences that we view a "nomad" science as eccentric and wholly different from the imperial. This is after all what we might think of the Greeks before Euclid, and unless we are careful, as a modern scientist these thoughts come naturally. However, it is often the case that state science retains of the nomad only what can be appropriated, with methods and motivations of mathematical pursuit conglomerated as less rigorous attempts - or even irrelevant according to our modern scientific understanding. In the philosopher Kuhn's Paradigm Theory, he points out that hypothesis and experimentation do not exist in isolation but are dependent on particular methodologies, terminologies, principles, tools, assumptions and axioms. This philosophy of scientific knowledge, although part of a larger argument for the basis of scientific knowledge, opens the eyes to the context given in any historical mathematical development.

Although at a first glance it appears that medieval Islam shares the same three modern reasons for the pursuit of mathematics, that is the pragmatic, the pedagogical and the aesthetic, the culture's answers would hold different significance. The importance of religious duty and the emphasis on the fulfilment of the soul in medieval Islamic culture produces a contrast against ours in a metaphysical dimension. This is how asking what mathematics means to us can shift a perspective.

Even in such a brief cross section through the history of something as seemingly simple as the number zero, our journey would need to span through various different geographical regions and time periods to get the full scope of the concept. In what this paper promised as a description of algebra in Islamic mathematics, it would be an incredibly difficult task to achieve without first looking at the knowledge available at the time, how it came to be and how it influenced further development. Maybe we are too concerned with pinpointing the conception of an idea that we fail to realise that perhaps the idea just existed in different forms. That could be something that we are missing from the teaching of western mathematics.

What we have learnt is that the motivation for what we produce and present is key to its understanding. The only limits are the attention span of your readers and the lens you want to shoot your subject in. Nonetheless, the course of history is the same.

Reference List

- Bawadi A., *Muhammad ibn Zakariya al-Razi* in M. Sharif (ed.), *A History of Muslim Philosophy*, Otto Harrassowitz, 1963], chapter XXII, pp421-433
- Berggren J.L., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*. Princeton University Press, 2007, p. 518.
- Boyer C., *Revival and Decline of Greek Mathematics*, 2nd Ed., New York, Wiley, 1991, p.180.
- Brentjes S., Morrison R. *The Sciences in Islamic Societies*, The New Cambridge History of Islam, 4, Cambridge: Cambridge University Press, 2010, p. 569.
- Chandler T., *'Four Thousand Years of Urban Growth: An Historical Census'*, Lewiston, NY: The Edwin Mellen Press, 1987.
- Cooke R., *The History of Mathematics: A Brief Course*, Wiley-Interscience, 1997, p. 167-168.
- Coolidge J.L., *The story of the binomial theorem*, The American Mathematical Monthly, Vol. 56, No. 3, 1949, p. 147–157.
- Derbyshire J., *Unknown Quantity: A Real And Imaginary History of Algebra*, Washington, DC, Joseph Henry Press, 2006, p. 35-36.
- Grimm R.E., *The Autobiography of Leonardo Pisano*, Fibonacci Quarterly, Vol. 11, No. 1, 1973, p. 99–104.
- Heine G., 'The Value of Mathematics - A Medieval Islamic View', in V. Katz (ed.), *Using History to Teach Mathematics: An International Perspective*, USA, The Mathematical Association of America, 2000, p.167.
- Jamil A., *The Determination of the Coordinates of Positions for the Correction of Distances between Cities - a Translation from the Arabic of al-Biruni's Kitab Tahdid Nihayat al-Amakin Litashih Masafat al-Masakin*, American University of Beirut, 1967
- Katz V.J., Barton B., *Stages in the History of Algebra with Implications for Teaching*, Educational Studies in Mathematics, Vol 66, No. 2, 2007, p. 185–201
- King D., *The Sacred Direction in Islam*, Interdisciplinary Science Reviews, Vol. 10, J W Arrowsmith, 1985, p.315.
- King D.A., "Mathematics applied to aspects of religious ritual in Islam". In I. Grattan-Guinness (ed.). *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, 1st Ed., JHU Press. 2003, p. 83.

- Macey S.L., *The Dynamics of Progress: Time, Method, and Measure*, Atlanta, Georgia: University of Georgia Press, , 1989, p. 92.
- Nasehpour P., *A Brief History of Algebra with a Focus on theDistributive Law and Semiring Theory* ,Golpayegan University of Technology, Iran, 2018, p. 2.
- Rafiabadi H.N., (ed.), Challenges to Religions and Islam: A Study of Muslim Movements, Personalities, Issues and Trends, Part 1, Sarup & Sons, 2007, p. 1141.
- Schappacher N., *Diophantus of Alexandria : a Text and its History (PDF)*,Institut de Recherche Mathématique Avancée, 2005, p. 18. Retrieved August 2021.
- Sesiano J., 'Islamic mathematics' in H. Selin(eds.), *Mathematics Across Cultures: The History of Non-Western Mathematics*, Springer, 2000, p. 148.
- Sigler L.E., *The English Edition of Fibonacci's Liber Abaci*, New York, Springer-Verlag, 2002.
- Smith C.D., *Developments in Secondary Mathematics.* Mathematics News Letter 5, no. 4, 1930, p. 4–8.

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