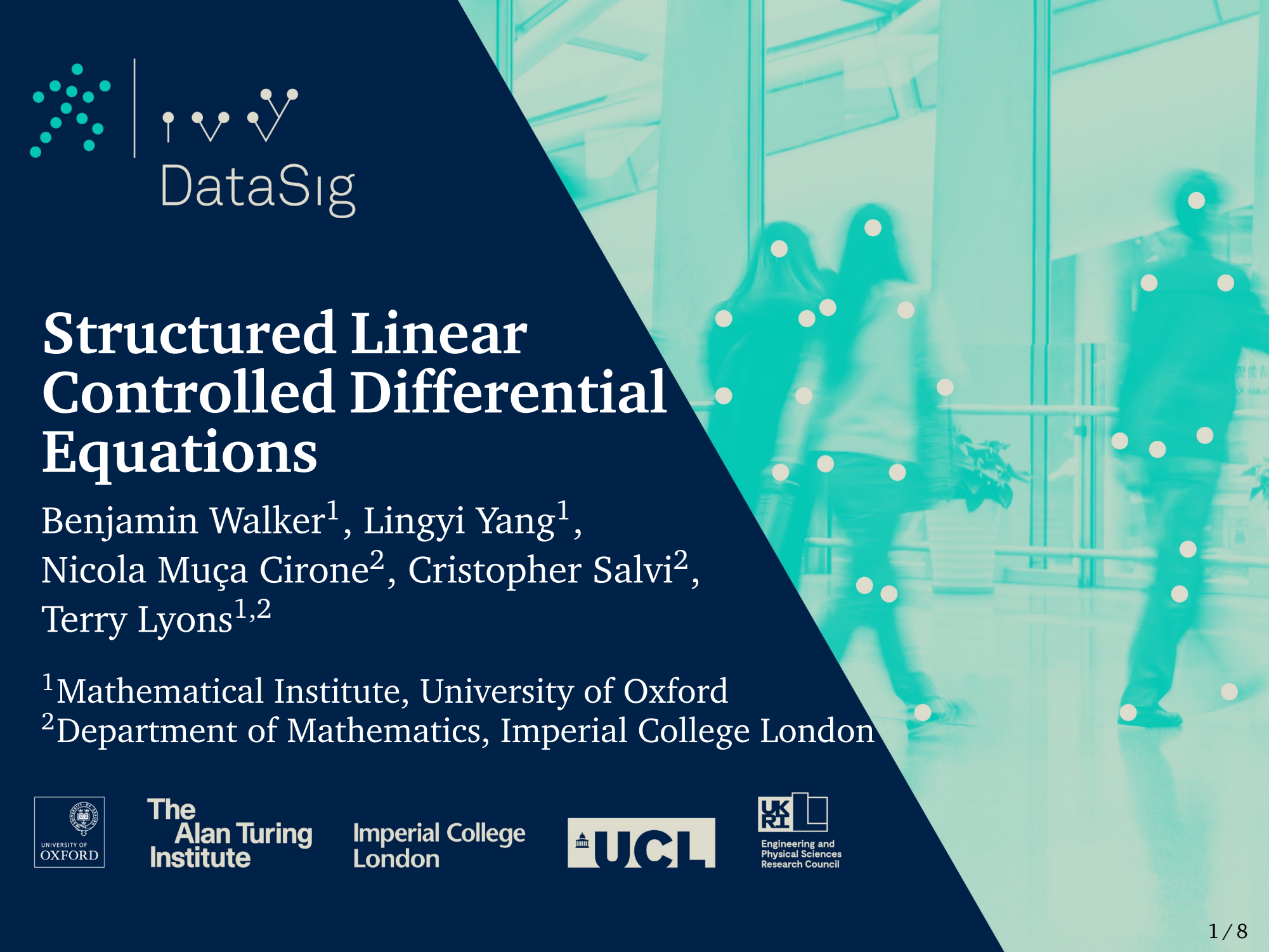

DataSig

Structured Linear Controlled Differential Equations

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Introduction

Structured linear controlled differential equations (SLiCEs) are a novel framework for understanding RNNs of the form:

$$h_{i+1} = A_\theta(x_i)h_i + B_\theta x_i,$$

where $x_i \in \mathbb{R}^{d_x}$, $h_i \in \mathbb{R}^{d_h}$, $A_\theta : \mathbb{R}^{d_x} \to \mathbb{R}^{d_h \times d_h}$, and $B_\theta \in \mathbb{R}^{d_h \times d_x}$.

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For more details see S. Yang et al. (2024b). "Parallelizing Linear Transformers with the Delta Rule over Sequence Length". In: *The Thirty-eighth Annual Conference on Neural Information Processing Systems*

Efficiency vs Expressivity

Why so many variants?

[1] N. M. Cirone et al. (2024). "Theoretical Foundations of Deep Selective State-Space Models". In: *Advances in Neural Information Processing Systems*

Efficiency vs Expressivity

Why so many variants?

- Dense matrices are maximally expressive but expensive[1].

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Is there a balance?

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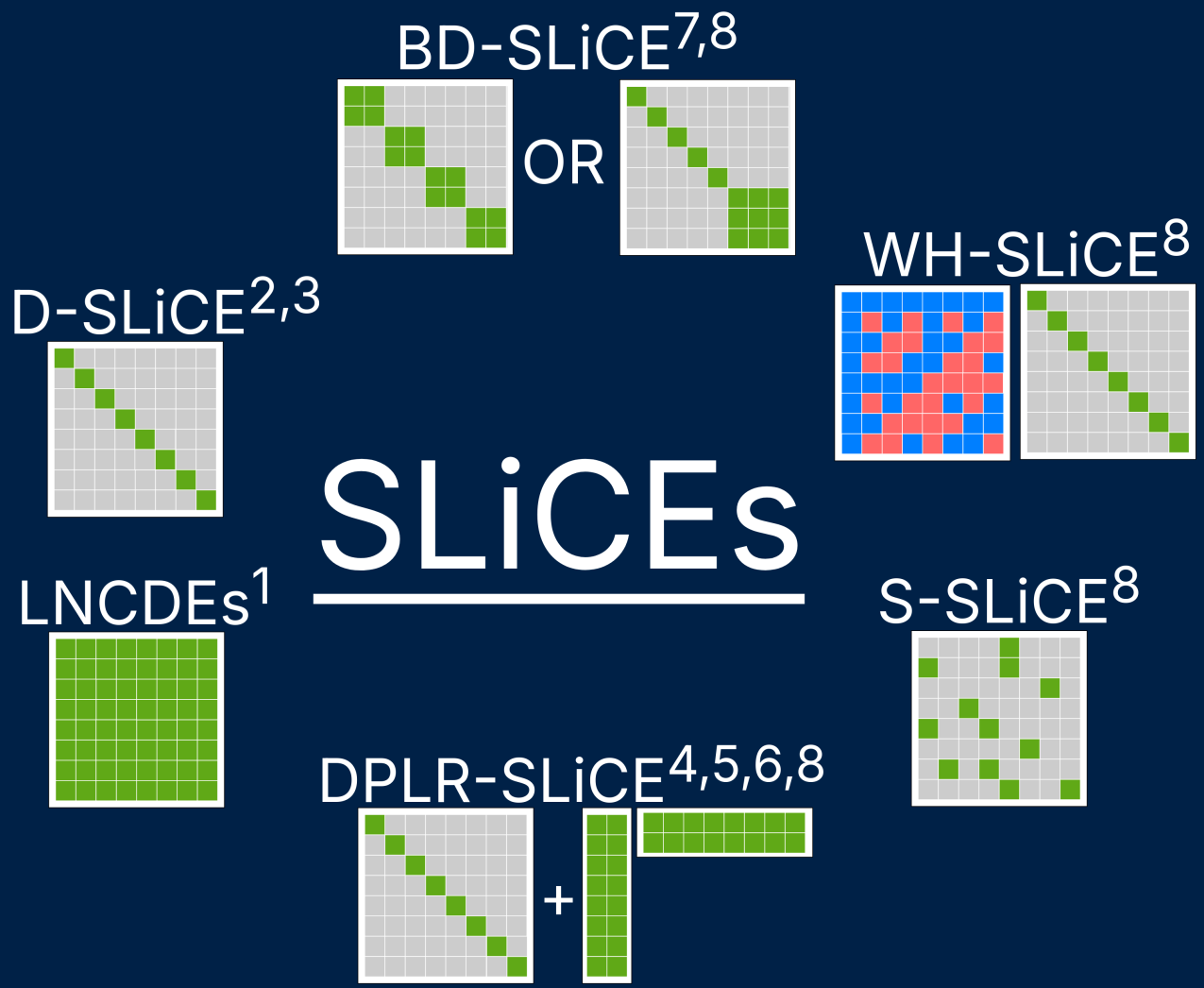

1.Cirone, N. M. et al. (2024). Theoretical foundations of deep selective state-space models. NeurIPS.

2. Gu, A. et al. (2022). Efficiently modeling long sequences with structured state spaces. ICLR.

3. Gu, A. et al. (2024). *Mamba: Linear-time sequence modeling with selective state spaces. COLM.*

4. Schlag, I. et al. (2021). Linear transformers are secretly fast weight programmers. ICML.

5. Yang, S. et al. (2024). Parallelizing linear transformers with the delta rule over sequence length. NeurIPS.

6. Siems, J. et al. (2025). DeltaProduct: Improving state-tracking in linear RNNs via Householder products. ICLR.

7. Fan, T. et al. (2024). Advancing regular language reasoning in linear recurrent neural networks. NAACL.

8. Walker, B. et al. (2025). Structured Linear CDEs: Maximally Expressive and Parallel-in-Time Models. NeurIPS.

SLiCE Comparison

Model	Parameters	Recurrent Cost	Parallel Cost	Maximally Expressive
DE-LNCDEs	$\mathcal{O}(d_h^3)$	$\mathcal{O}(nd_h^3)$	$\mathcal{O}(\log(n), d_h^3)$	Yes

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D-SLiCEs	$\mathcal{O}(d_h^2)$	$\mathcal{O}(nd_h^2)$	$\mathcal{O}(\log(n), d_h^2)$	No
DPLR-SLiCEs	$\mathcal{O}(rd_h^2)$	$\mathcal{O}(nrd_h^2)$	$\mathcal{O}(\log(n), d_h^3)$	Yes
S–SLiCEs	$\mathcal{O}(d_h^{2+\epsilon})$	$\mathcal{O}(nd_h^{2+\epsilon})$	$\mathcal{O}(\log(n), d_h^3)$	Yes
WH–SLiCEs	$\mathcal{O}(d_h^2)$	$\mathcal{O}(nd_h^2)$	$\mathcal{O}(\log(n), d_h^3)$	Yes
BD–SLiCEs	$\mathcal{O}\left(d_h \sum_j b_j^2\right)$	$\mathcal{O}\left(nd_h \sum_j b_j^2\right)$	$\mathcal{O}\left(\log(n), d_h \sum_j b_j^2\right)$	Yes

Table 1: Comparison of SLiCEs on parameter count, computational cost, and expressivity. Parallel cost is measured as $\mathcal{O}$(scan depth, cost per composition) when applying a parallel associative scan.

Length Generalisation

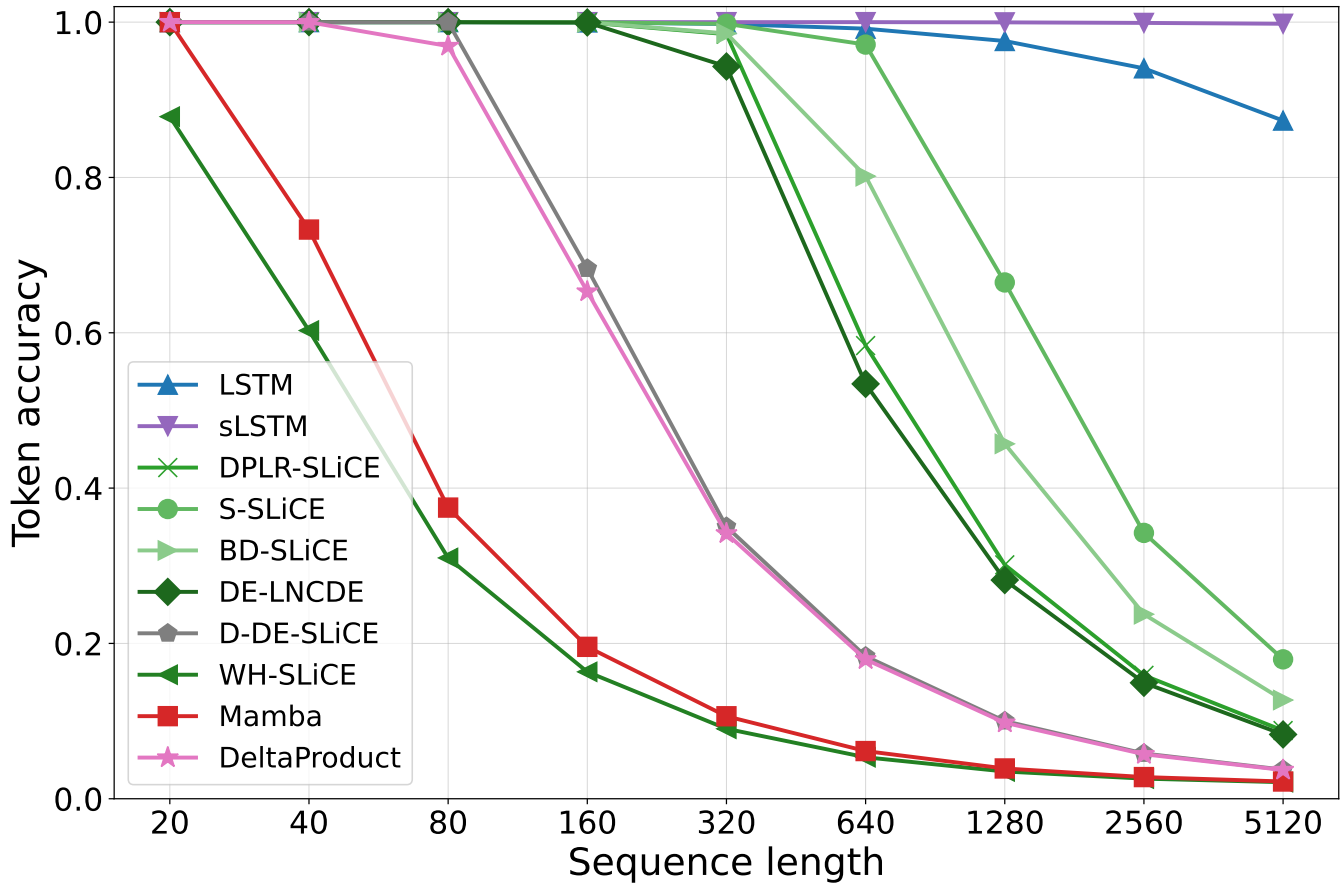

Figure 1: Length generalisation on the A_5 benchmark.

Time Series Classification

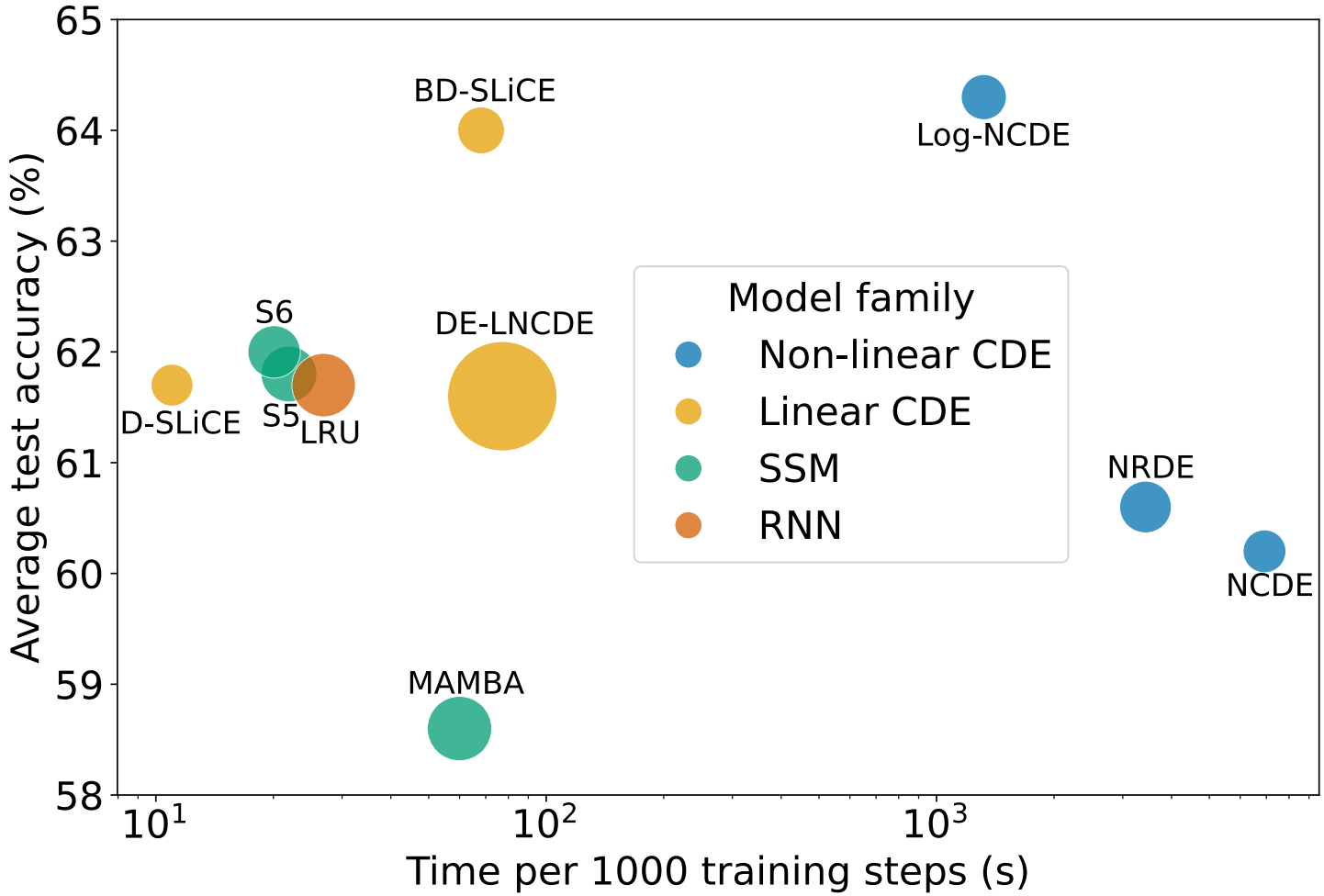

Figure 2: Accuracy, speed, and memory footprint on the UEA-MTSCA.

Thank you!

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